

# THE FREE LOOP SPACE HOMOLOGY OF $(n-1)$ -CONNECTED $2n$ -MANIFOLDS

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ABSTRACT. Our goal in this paper is to compute the integral free loop space homology of  $(n-1)$ -connected  $2n$ -manifolds  $M$ ,  $n \geq 2$ . We do this when  $n \neq 2, 4, 8$ , or when  $n \neq 2$  and  $\tilde{H}^*(M)$  has trivial cup product squares, though the techniques used here should extend to a much wider range of manifolds. We also give partial information concerning the action of the Batalin-Vilkovisky operator.

## 1. INTRODUCTION

Let  $\mathcal{L}X = \text{map}(S^1, X)$  denote the free loop space on  $X$ . This space comes equipped with an action  $\nu: S^1 \times \mathcal{L}X \rightarrow \mathcal{L}X$  that rotates loops, and an induced degree 1 homomorphism

$$\Delta: H_*(\mathcal{L}X) \rightarrow H_{*+1}(\mathcal{L}X)$$

known as the *BV-operator*, defined by setting  $\Delta(a) = \nu_*([S^1] \otimes a)$ . In addition Chas and Sullivan [9] constructed a pairing

$$H_p(\mathcal{L}X) \otimes H_q(\mathcal{L}X) \rightarrow H_{p+q-d}(\mathcal{L}X)$$

on a closed oriented  $d$ -manifold  $X$  that (together with the *BV-operator*) turns the shifted homology  $\mathbb{H}_*(\mathcal{L}X) = H_{*+d}(\mathcal{L}X)$  into a Batalin-Vilkovisky (BV)-algebra.

Batalin-Vilkovisky algebras have been computed in only a few special cases. One of the more general results to date (due to Felix and Thomas [13]) states that over a field  $F$  of characteristic zero and 1-connected  $X$ ,  $\mathbb{H}_*(\mathcal{L}X; F)$  is isomorphic to a BV-algebra structure defined on the Hochschild cohomology  $HH^*(C^*(X), C^*(X))$ . Unfortunately, this theorem is generally not true for fields with nonzero characteristic [21]. Beyond these results, the BV-algebra over various coefficient rings has been completely determined for spheres [10, 26, 21], certain Stiefel manifolds [25], Lie groups [18], and projective spaces [29, 23, 10, 28, 17], using a mixture of techniques ranging from homotopy theoretic to geometric, as well as the well-known connections to Hochschild cohomology.

In this paper we focus on the free loop space homology of highly connected  $2n$ -manifolds, together with the action of the BV-operator. The coefficient ring  $R$  for homology and cohomology is assumed to be either any field, or the integers  $\mathbb{Z}$ , but we suppress it from notation most of the time. Fix  $n \geq 2$ ,  $M$  a  $(n-1)$ -connected, closed, oriented  $2n$ -manifold with  $H^n(M)$  of rank  $m \geq 1$ . Let

$$C = [c_{ij} = \langle a_i \cup a_j, [M] \rangle]$$

be the  $m \times m$  matrix for the intersection form  $H^n(M) \times H^n(M) \rightarrow \mathbb{Z}$  with respect to some choice of basis  $\{a_1, \dots, a_m\}$  for  $H^n(M)$  (we use the same notation for the dual basis of  $H^n(M)$ ). This form is nonsingular, symmetric when  $n$  is even, and skew-symmetric when  $n$  is odd.

Denote  $H^n(M)$  and  $H^{2n}(M) \cong \mathbb{Z}$  by the free graded modules  $R$ -modules  $A = R\{a_1, \dots, a_m\}$  and  $K = R\{[M]\}$ , and the desuspension of  $A$  by  $V = R\{u_1, \dots, u_m\}$  with  $|u_i| = n-1$ . Let

$$T(V) = R \oplus \bigoplus_{i \geq 1} V^{\otimes i}$$

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be the free tensor algebra generated by  $V$ , and  $I$  be the two-sided ideal of the tensor algebra  $T(V)$  generated by the following degree  $2n - 2$  element

$$\chi = \sum_{i < j} c_{ij} [u_i, u_j] + \sum_i c_{ii} u_i^2,$$

where  $[x, y] = xy - (-1)^{|x||y|}yx$  denotes the graded Lie bracket in  $T(V)$ . Take the quotient algebra

$$U = \frac{T(V)}{I}$$

and the degree  $-1$  maps of graded  $R$ -modules  $d: A \otimes U \rightarrow U$  and  $d': K \otimes U \rightarrow A \otimes U$ , which are given for any  $y \in U$  by the formulas

$$d(a_i \otimes y) = [u_i, y]$$

$$d'([M] \otimes y) = \sum_{i,j} c_{ij} (a_j \otimes [u_i, y]).$$

If we apply the Jacobi identity to the summands  $c_{ij}(a_j \otimes [u_i, y])$  in  $d \circ d'(y)$  for  $i < j$  (keeping in mind that  $c_{ij} = (-1)^n c_{ji}$ ,  $[u_i, [u_i, y]] = [u_i^2, y]$ , and that products with  $\chi$  are identified with zero in  $U$ ), we see that  $\text{Im } d' \subseteq \ker d$ , so we obtain a chain complex

$$0 \rightarrow K \otimes U \xrightarrow{d'} A \otimes U \xrightarrow{d} U \rightarrow 0.$$

Now take the homology of this chain complex. That is, take the following graded  $R$ -modules:

$$\mathcal{Q} = \frac{U}{\text{Im } d}, \quad \mathcal{W} = \frac{\ker d}{\text{Im } d'}, \quad \mathcal{Z} = \ker d'.$$

One can think of  $\mathcal{W}$  by first taking the  $R$ -submodule  $W'$  of  $\Sigma^{-1}A \otimes T(V) \cong T(V)$  generated by elements that are invariant modulo  $I$  under graded cyclic permutations, that is, invariant after projecting to  $U$ . Then  $\mathcal{W}$  is the projection of  $\Sigma W'$  onto  $(A \otimes U)/\text{Im } d'$ .

Our main result is that the homology of this chain complex is the integral free loop space homology of  $M$  under some conditions:

**Theorem 1.1.** *Suppose  $n \geq 2$  and  $\text{rank } H^n(M) \geq 1$ . If  $n \neq 2, 4, 8$ , or  $n \neq 2$  and  $\tilde{H}^*(M)$  has trivial cup product squares, then there exists an isomorphism of graded  $R$ -modules*

$$H_*(\mathcal{L}M) \cong \mathcal{Q} \oplus \mathcal{W} \oplus \mathcal{Z}.$$

The restriction away from 2, 4, and 8 traces back to an argument that we use to determine  $H_*(\Omega M)$ , which does not apply to situation where there are cup product squares equal to the fundamental class  $[M]$ , or  $-[M]$ . Failure of a degree placement argument to compute certain differentials is another reason that we restrict away from  $n = 2$ .

We also determine the action of the BV-operator on  $H_*(\mathcal{L}M; \mathbb{Q})$ , in a sense, up-to-abelianization of  $U$  when  $n > 3$  is odd.

Consider the graded abelianization map  $T(V) \xrightarrow{\eta} S(V)$ , where  $S(V)$  is the free graded symmetric algebra generated by  $V$ . Since  $\eta(\chi) = 0$ ,  $\eta$  factors through  $U \xrightarrow{\eta} S(V)$ . Also, consider the maps  $A \otimes U \xrightarrow{\mathbb{1}_A \otimes \eta} A \otimes S(V)$  and  $K \otimes U \xrightarrow{\mathbb{1}_K \otimes \eta} K \otimes S(V)$ . Since  $(\mathbb{1}_A \otimes \eta) \circ d' = 0$  and  $\eta \circ d = 0$ , then  $\eta$  and these two maps induce *abelianization* maps

$$\mathcal{Q} \xrightarrow{\eta_q} S(V),$$

$$\mathcal{W} \xrightarrow{\eta_w} A \otimes S(V),$$

$$\mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V).$$

**Theorem 1.2.** *Let  $n > 3$  be odd. The BV operator  $\Delta: H_*(\mathcal{L}M; \mathbb{Q}) \rightarrow H_{*+1}(\mathcal{L}M; \mathbb{Q})$  satisfies  $\Delta(\mathcal{Q}) \subseteq \mathcal{W}$  and  $\Delta(\mathcal{W}) \subseteq \mathcal{Z}$ , and  $\Delta(\mathcal{Z}) = \{0\}$ . Moreover, the composite  $\mathcal{Q} \xrightarrow{\Delta} \mathcal{W} \xrightarrow{\eta_w} A \otimes S(V)$  is given by*

$$\eta_w \circ \Delta(1 \otimes (u_{i_1} \cdots u_{i_k})) = \sum_{j=1}^k a_{i_j} \otimes (u_{i_1} \cdots u_{i_{j-1}} u_{i_{j+1}} \cdots u_{i_k}),$$

and  $\mathcal{W} \xrightarrow{\Delta} \mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V)$  is the restriction to  $\ker d$  of the map  $(A \otimes U)/\text{Im } d' \rightarrow S(A) \otimes S(V)$  given by

$$a_i \otimes (u_{i_1} \cdots u_{i_k}) \mapsto \sum_{j=1}^k a_i a_{i_j} \otimes (u_{i_1} \cdots u_{i_{j-1}} u_{i_{j+1}} \cdots u_{i_k}),$$

where  $[M] \in K$  is identified with  $(\sum_{i \leq j} c_{ij} a_i a_j) \in S(A)$ .

Berglund and Börjeson [6] have subsequently computed the free loop space homology of highly connected manifolds (including the ones considered in this paper) using different techniques. They also give a description of the action of the BV-operator and the Chas-Sullivan loop product. With a bit of effort it is likely that the spectral sequence methods in this paper can be extended to cover many of the highly connected manifolds in [6]. For example, the based loop space homology of highly connected manifolds is largely known [5], and this is one of the main ingredients used in our calculations. On the other hand, we do not know whether a complete description of the Chas-Sullivan loop product and BV-operator is possible using our approach - one difficulty being extension issues in the Cohen-Jones-Yan spectral sequence [11] when computing the loop product, together with a seeming incompatibility between the BV-operator and the Serre spectral sequence of a free loop fibration.

We should mention that there are sources of application for the above calculations that go beyond the classical question: *are there infinitely many geometrically distinct periodic geodesics on a Riemannian manifold  $M$ ?* For example, detailed information about the Betti numbers of  $\mathcal{L}M$  reflects more detailed information about the number of geodesics of variable length. See [3, 6, 2, 14].

## 2. A USEFUL LEMMA

Take a fibration sequence  $F \xrightarrow{i} X \xrightarrow{f} B$  with  $B$  simply-connected. Recall the induced homotopy fibration sequence

$$(1) \quad \Omega B \xrightarrow{\vartheta} F \xrightarrow{i} X$$

is a *principal* homotopy fibration. Namely, there is a homotopy associative  $H$ -space structure on the homotopy fiber  $\Omega B$  together with a left action

$$\theta: \Omega B \times F \rightarrow F$$

that fits into a homotopy commutative square

$$\begin{array}{ccc} \Omega B \times \Omega B & \xrightarrow{1 \times \vartheta} & \Omega B \times F \\ \text{mult.} \downarrow & & \downarrow \theta \\ \Omega B & \xrightarrow{\vartheta} & F. \end{array}$$

In our case the  $H$ -space multiplication *mult.* on  $\Omega B$  is taken as the one defined by composing loops, and the action  $\theta$  is defined by applying the homotopy lifting property to loops in  $B$ .

By a result of Moore [22], the homology Serre spectral sequence  $\xi$  of a principal fibration such as (1) has a left  $H_*(\Omega B)$ -module induced by the associated action  $\theta$ . Namely, there is a left action  $H_*(\Omega B) \otimes \xi_{i,j}^r \rightarrow \xi_{i,j+*}^r$  reducing to the Pontrjagin multiplication on  $\xi_{0,*}^2 \cong H_*(\Omega B)$  and differentials respect this action. Most of the effort in computing differentials is therefore reduced to determining those emanating from the degree 0 horizontal line.

Since fibrations are characterized by the homotopy lifting property, one might also expect  $\theta$  to have a direct bearing on the homology Serre spectral sequence for our original fibration  $f$ . This was exploited by McCleary in [20], where he used a result of Brown [8] and Shih [24] to give a computation of the free loop space homology of certain low rank Stiefel manifolds. The following proposition strengthens the result in [8, 24] by doing away with an assumption about certain elements being transgressive. The proof is moreover fairly simple. Let

$$\mathcal{E} = \{\mathcal{E}^r, \delta^r\}$$

denote the homology Serre spectral sequence for  $f$ , and

$$E = \{E^r, d^r\}$$

the homology Serre spectral sequence for the path-loop fibration sequence  $\Omega B \xrightarrow{c} \mathcal{P}B \xrightarrow{ev_1} B$ .

**Proposition 2.1.** *Suppose  $H_*(B)$  and  $H_*(F)$  are torsion free. Given  $z \in H_*(B)$ , and  $\sum_i x_i \otimes v_i \in E_{*,*}^2 \cong H_*(B) \otimes H_*(\Omega B)$ , suppose  $d^s(z \otimes 1) = d^s(\sum_i x_i \otimes v_i) = 0$  in  $E_{*,*}^s$  for  $2 \leq s < r$ , and*

$$d^r(z \otimes 1) = \sum_i x_i \otimes v_i.$$

*Then given  $z \otimes y \in \mathcal{E}_{*,*}^2 \cong H_*(B) \otimes H_*(F)$  for any  $y \in H_*(F)$ , for each  $2 \leq s < r$  we have*

$$\delta^s(z \otimes y) = \delta^s(\sum_i x_i \otimes \theta_*(v_i \otimes y)) = 0$$

and

$$\delta^r(z \otimes y) = \sum_i x_i \otimes \theta_*(v_i \otimes y).$$

*Proof.* First recall the following well-known property (which is essentially the homotopy lifting property in disguise). Let  $P^{ev_0, f} \subseteq \text{map}([0, 1], B) \times X$  be the pullback of  $X \xrightarrow{f} B$  and the evaluation map  $\text{map}([0, 1], B) \xrightarrow{ev_0} B$ , where  $ev_t(\omega) = \omega(t)$ . Now consider the map  $\bar{f}: \text{map}([0, 1], X) \rightarrow P^{ev_0, f}$  defined by  $\bar{f}(\omega) = (f \circ \omega, \omega(0))$ . Then a surjection  $f$  is a fibration if and only if there exists a map  $g: P^{ev_0, f} \rightarrow \text{map}([0, 1], X)$  such that  $\bar{f} \circ g = \mathbb{1}: P^{ev_0, f} \rightarrow P^{ev_0, f}$ .

Take the inclusion  $\phi: \mathcal{P}B \times F \rightarrow P^{ev_0, f}$  given by  $\phi(\omega, a) = (\omega, a)$ , and take the composite

$$\bar{\theta}: (\mathcal{P}B \times F) \xrightarrow{\phi} P^{ev_0, f} \xrightarrow{g} \text{map}([0, 1], X) \xrightarrow{ev_1} X.$$

Let the fibration sequence

$$(2) \quad \Omega B \times F \xrightarrow{c \times \mathbb{1}} \mathcal{P}B \times F \xrightarrow{ev_1 \times * } B \times *$$

be the product of the path-loop fibration sequence  $\Omega B \xrightarrow{c} \mathcal{P}B \xrightarrow{ev_1} B$  and the trivial fibration sequence  $F \xrightarrow{\mathbb{1}} F \xrightarrow{*} *$ . Let  $E = \{E^s, d^s\}$  and  $\dot{E} = \{\dot{E}^s, \dot{d}^s\}$  be the homology Serre spectral sequences for the path-loop and trivial fibration respectively, and  $\hat{E} = \{\hat{E}^s, \hat{d}^s\}$  be the homology spectral sequence for their product (2). Define a differential  $d_{\otimes}^s: E^s \otimes \dot{E}^s \rightarrow E^s \otimes \hat{E}^s$  by  $\hat{d}^s(a \otimes b) = (d^s(a) \otimes b) + (-1)^{|a|}(a \otimes \dot{d}^s(b))$ . Since  $H_*(F)$  is torsion-free,  $\hat{E}^s = E^s \otimes \dot{E}^s$  and  $\hat{d}^s = d_{\otimes}^s$  (see [7, 15]). In our case  $\dot{d} = 0$ , so we have

$$\hat{d}^s(a \otimes b) = d^s(a) \otimes b$$

for any  $a \in E^s$  and  $b \in \dot{E}^s$ . One can easily check that the following diagram of fibration sequences commutes

$$(3) \quad \begin{array}{ccccc} \Omega B \times F & \xrightarrow{c \times \mathbb{1}} & \mathcal{P}B \times F & \xrightarrow{ev_1 \times * } & B \times * \\ \downarrow \theta & & \downarrow \bar{\theta} & & \parallel \\ F & \xrightarrow{i} & X & \xrightarrow{f} & B, \end{array}$$

with our action  $\theta$  being in fact the restriction of  $\bar{\theta}$  to the subspace  $\Omega B \times F$ . Let

$$\zeta: \hat{E} = E \otimes \mathring{E} \longrightarrow \mathcal{E}$$

be the morphism of spectral sequences induced by this diagram.

Since  $d^s(z \otimes 1) = 0 \in E_{*,*}^s$  for  $2 \leq s < r$  and  $d^r(z \otimes 1) = \sum_i x_i \otimes v_i$ , then for any  $b \in \mathring{E}^s$

$$\hat{d}^s((z \otimes 1) \otimes b) = d^s(z \otimes 1) \otimes b = 0$$

$$\hat{d}^r((z \otimes 1) \otimes b) = d^r(z \otimes 1) \otimes b = \sum_i (x_i \otimes v_i) \otimes b,$$

which we use to obtain

$$\begin{aligned} \delta^r(z \otimes y) &= \delta^r(\zeta^r((z \otimes 1) \otimes (1 \otimes y))) \\ &= \zeta^r(\hat{d}^r((z \otimes 1) \otimes (1 \otimes y))) \\ &= \zeta^r\left(\sum_i (x_i \otimes v_i) \otimes (1 \otimes y)\right) \\ &= \sum_i x_i \otimes \theta_*(v_i \otimes y), \end{aligned}$$

and similarly,  $\delta^s(z \otimes y) = 0$  for  $2 \leq s < r$ .

In a similarly manner, we see  $\hat{d}^s((\sum_i x_i \otimes v_i) \otimes b) = 0$  for  $2 \leq s < r$  and (in turn)  $\delta^s(\sum_i x_i \otimes \theta_*(v_i \otimes y)) = 0$  using the fact that  $d^s(\sum_i x_i \otimes v_i) = 0$  (so the above equations make sense).  $\square$

We now turn our attention towards the free loop space fibration sequence

$$(4) \quad \Omega B \xrightarrow{\vartheta} \mathcal{L}B \xrightarrow{ev_1} B.$$

The map  $\vartheta$  is the canonical inclusion  $\Omega B \subseteq \mathcal{L}B$ , and  $ev_1$  is the evaluation map  $ev_1(\omega) = \omega(1)$ . The homology Serre spectral sequence for this fibration sequence will be denoted by

$$\mathcal{E} = \{\mathcal{E}^r, \delta^r\},$$

and as before  $E = \{E^r, d^r\}$  is the homology Serre spectral sequence for the path-loop fibration of  $B$ . The path-loop fibration is principal, so  $E$  has a left  $H_*(\Omega B)$ -module as described before which the differentials  $d$  respect.

Some basic properties of the free loop space fibration are as follows. The map  $\mathcal{L}B \xrightarrow{ev_1} B$  has a section  $B \xrightarrow{s} \mathcal{L}B$  defined by mapping a point  $b \in B$  to the constant loop at  $b$ , which implies the connecting map  $\varrho$  for the induced principal homotopy fibration  $\Omega B \xrightarrow{\varrho} \Omega B \xrightarrow{\vartheta} \mathcal{L}B$  is null homotopic. The associated left action

$$\theta: \Omega B \times \Omega B \longrightarrow \Omega B$$

is given by

$$\theta(\omega, \lambda) = \omega \cdot \lambda \cdot \omega^{-1}$$

for any  $\omega, \lambda \in \Omega B$ . If  $v \in H_*(\Omega B)$  is primitive, then for any  $y \in H_*(\Omega B)$  one has the formula

$$\theta_*(v \otimes y) = (-1)^{|v||y|} yv - vy = -[v, y],$$

where the multiplication on  $H_*(\Omega B)$  is the Pontrjagin multiplication induced by loop composition on  $\Omega B$ . The proof of these can be found in [20] for example. Combining these properties with Propositions 2.1 gives the following description of the differentials in the spectral sequence  $\mathcal{E}$ .

**Proposition 2.2.** *Suppose  $H_*(B)$  and  $H_*(\Omega B)$  are torsion free, and  $B$  is 1-connected. Given  $z \in H_*(B)$ , and  $\sum_i x_i \otimes v_i \in E_{*,*}^2$  with  $v_i$  primitive in  $H_*(\Omega B)$ , suppose that  $d^s(z \otimes 1) = 0$  and  $d^s(\sum_i x_i \otimes v_i) = 0$  in  $E_{*,*}^s$  for  $2 \leq s < r$ , and*

$$d^r(z \otimes 1) = \sum_i x_i \otimes v_i.$$

*Then given  $z \otimes y \in \mathcal{E}_{*,*}^2$  for any  $y \in H_*(\Omega B)$ , for each  $2 \leq s < r$  we have*

$$\delta^s(z \otimes y) = \delta^s(\sum_i x_i \otimes [v_i, y]) = 0$$

*and*

$$\delta^r(z \otimes y) = -\sum_i x_i \otimes [v_i, y].$$

□

There are instances where this formula fails to give us enough information to determine some of the higher differentials. For example, if we found ourselves in the situation where  $\delta^s(z \otimes y) = 0$  for  $s \leq r$  and  $d^r(z \otimes y) \neq 0$ , then  $z \otimes y \in \mathcal{E}_{*,*}^r$  survives to the  $\mathcal{E}^{r+1}$  page, while  $z \otimes y$  is not an element in  $E_{*,*}^{r+1}$ . In such case  $\delta^s(z \otimes y)$  remains mysterious when  $s > r$ . An example where this situation happens in practice is the case of 4-manifolds omitted from Theorem 1.1.

### 3. BASED LOOP SPACE HOMOLOGY

Returning to our  $2n$ -manifold  $M$  in the introduction, we consider the Hopf algebra  $H_*(\Omega M)$ . This is the last piece in the puzzle required to prove Theorem 1.1. By Poincaré duality the only nonzero reduced homology groups of  $M$  are in degrees  $n$  and  $2n$ . This implies  $M$  has a cell decomposition given by attaching an  $n$ -cell to an  $m$ -fold wedge of  $n$ -spheres  $\bigvee_m S^n \simeq M - *$ , where  $m = \text{rank } H^n(M)$ .

Generally, if a space  $Y$  is formed by attaching a  $k$ -cell to a space  $X$  via an attaching map  $S^{k-1} \xrightarrow{\alpha} X$ , and  $\alpha'$  is its adjoint, the composite with the looped inclusion  $S^{k-2} \xrightarrow{\alpha'} \Omega X \xrightarrow{\Omega i} \Omega Y$  is nullhomotopic, so one obtains a factorization of Hopf algebras through Hopf algebra maps

$$(5) \quad \begin{array}{ccc} & H_*(\Omega X; R)/I & \\ & \nearrow & \downarrow \theta \\ H_*(\Omega X; R) & \xrightarrow{(\Omega i)_*} & H_*(\Omega Y; R), \end{array}$$

where  $I$  is the two-sided ideal generated by  $\alpha'([S^{k-2}]) \in H_{k-2}(\Omega X; R)$ . The problem of determining the conditions under which  $\theta$  is a Hopf algebra isomorphism is part of what is known as the *cell-attachment problem*. One of these conditions - the *inert condition* - states somewhat suprisingly that  $\theta$  is a Hopf algebra isomorphism when  $R$  is a field if and only if  $(\Omega i)_*$  is a surjection ([19, 16, 12]). Here we select  $k = 2n$ ,  $Y \simeq M$ , and  $X \simeq M - *$ , and use the inert condition to prove the following:

**Proposition 3.1.** *Suppose  $n \geq 2$  and  $\text{rank } H^n(M) \geq 1$ . If  $n \neq 2, 4, 8$ , or  $\tilde{H}^*(M)$  has trivial cup product squares,*

- (i) *There is an isomorphism of Hopf algebras (free as  $R$ -modules)*

$$H_*(\Omega M) \cong \frac{T(V)}{I}$$

*where  $V = R\{u_1, \dots, u_m\}$ ,  $|u_i| = n - 1$ .*

- (ii) *The element  $\alpha'_*([S^{2n-2}])$  generating the two-sided ideal  $I$  is given by*

$$\alpha'_*([S^{2n-2}]) = \sum_{i < j} c_{ij} [u_j, u_i] + \sum_i c_{ii} u_i^2.$$

*Proof of part (i).* In [4],  $\Omega M$  is shown to be a homotopy retract of  $\Omega(M - *)$  when  $n \neq 2, 4, 8$ , or when  $\tilde{H}^*(M)$  has trivial cup product squares. Therefore  $(\Omega i)_*$  is a split epimorphism, so we obtain  $H_*(\Omega M; F) \cong H_*(\Omega(M - *); F)/I$  for any field  $F$ . Moreover, since  $M - *$  is homotopy equivalent to  $\bigvee_m S^n$ , the  $\mathbb{Z}$ -module  $H_*(\Omega(M - *); \mathbb{Z}) \cong T(V)$  is torsion-free. Therefore  $H_*(\Omega M; \mathbb{Z})$  is torsion-free, and the Hopf algebra isomorphism holds for  $R = \mathbb{Z}$  as well.  $\square$

*Proof of part (ii).* We will write  $u_j = (\Omega i)_*(u_j) \in H_{n-1}(\Omega M)$ , and take  $u_j$  to be the transgression of  $a_j \in H_n(M)$ .

Since the elements  $u_1, \dots, u_m$  in  $H_{n-1}(\Omega(M - *))$  are primitive, and there are no monomials of length greater than 2 in degree  $2n - 2$ , the elements  $u_i^2$  and  $[u_j, u_i]$  form a basis for the primitives in  $H_{2n-2}(\Omega(M - *))$ . Now  $\alpha'_*([S^{2n-2}])$  is primitive since  $[S^{2n-2}]$  is primitive, so we can set

$$(\alpha')_*([S^{2n-2}]) = \sum_{i < j} c''_{ij} [u_i, u_j] + \sum_i c''_{ii} u_i^2$$

for some integers  $c''_{ij}$ .

Consider the homology Serre spectral sequence  $E = (E^r, d^r)$  for the (principal) path-loop fibration sequence  $M$ , with

$$E_{*,*}^2 = H_*(M) \otimes H_*(\Omega M).$$

On the dual cohomology spectral sequence we have the formula

$$d_n(a_j \otimes u_i) = d_n(a_j \otimes 1)(1 \otimes u_i) + (-1)^n (a_j \otimes 1) d_n(1 \otimes u_i) = (-1)^n (a_j \otimes 1)(a_i \otimes 1) = c_{ij}([M]^* \otimes 1),$$

so dualizing back to the homology spectral sequence gives us

$$(6) \quad d^n([M] \otimes 1) = \sum_{i,j} c_{ij} (a_j \otimes u_i).$$

Take  $\bar{E} = (\bar{E}^r, \bar{d}^r)$  to be the homology Serre spectral sequence for the path-loop fibration of  $M - *$ . The inclusion  $(M - *) \rightarrow M$  induces an inclusion of the corresponding path-loop fibrations of  $(M - *)$  and  $M$ , and in turn a morphism of spectral sequences  $\gamma: \bar{E} \rightarrow E$ . On the second page of spectral sequences  $\gamma_2$  maps  $1 \otimes u_i$  to  $1 \otimes u_i$  and  $a_i \otimes 1$  to  $a_i \otimes 1$ , and  $\bar{E}_{n,n-1}^r \xrightarrow{\gamma_r} E_{n,n-1}^r$  is an isomorphism for  $2 \leq r \leq n$ .

By part (i) of the theorem (and preceding discussion),  $(\alpha')_*([S^{2n-2}])$  generates the kernel of  $(\Omega i)_*: H_{2n-2}(\Omega(M - *)) \rightarrow H_{2n-2}(\Omega M)$ , so  $1 \otimes (\alpha)_*([S^{2n-2}])$  generates the kernel of  $\gamma_2: E_{0,2n-2}^2 \rightarrow \bar{E}_{0,2n-2}^2$ . Since  $\gamma_r: \bar{E}_{i,j}^r \rightarrow E_{i,j}^r$  is an isomorphism for  $i < n, j < 2n - 2$ , and all  $r$ , then in fact  $1 \otimes (\alpha')_*([S^{2n-2}])$  generates the kernel of the map  $\bar{E}_{0,2n-2}^r \xrightarrow{\gamma_r} E_{0,2n-2}^r$  for  $2 \leq r \leq n$ .

Take the element

$$\zeta'' = \sum_{i \leq j} c''_{ij} (a_j \otimes u_i - a_i \otimes u_j)$$

in  $\bar{E}_{n,n-1}^r$ , for  $2 \leq r \leq n$ . Then

$$(7) \quad \gamma_n(\zeta'') = \sum_{i \leq j} c''_{ij} (a_j \otimes u_i - a_i \otimes u_j),$$

and in  $\bar{E}_{0,2n-2}^n$  we have

$$1 \otimes (\alpha')_*([S^{2n-2}]) = \sum_{i \leq j} c''_{ij} (1 \otimes [u_i, u_j]) = \bar{d}^n(\zeta'').$$

Since  $\bar{E}_{i,j}^r = \{0\}$  for  $i > n$  and  $\bar{E}_{*,*}^\infty = \{0\}$ , the differential  $\bar{E}_{n,n-1}^n \xrightarrow{\bar{d}^n} \bar{E}_{0,2n-2}^n$  is an isomorphism, and since  $\bar{E}_{n,n-1}^n \xrightarrow{\gamma_n} E_{n,n-1}^n$  is an isomorphism and  $1 \otimes (\alpha')_*([S^{2n-2}])$  generates the kernel of  $\bar{E}_{0,2n-2}^n \xrightarrow{\gamma_n} E_{0,2n-2}^n$ , by naturality we see that the kernel of the differential  $E_{n,n-1}^n \xrightarrow{d^n} E_{0,2n-2}^n$  is generated by  $\gamma_n(\zeta'')$ . In particular, we may project  $\gamma_n(\zeta'')$  down to  $E_{*,*}^\infty$ .

Let

$$\mathcal{I} = \text{Im } d^n : E_{2n,0}^n \longrightarrow E_{n,n-1}^n$$

$$\mathcal{K} = \ker d^n : E_{n,n-1}^n \longrightarrow E_{0,2n-2}^n.$$

As we saw above,  $\mathcal{I}$  is generated by  $d^n([M] \otimes 1)$ , and  $\gamma_n(\zeta'')$  generates  $\mathcal{K}$ . But the short exact sequence

$$0 \longrightarrow E_{2n,0}^n \xrightarrow{d^n} E_{n,n-1}^n \xrightarrow{d^n} E_{0,n-2}^n \longrightarrow 0$$

implies  $\mathcal{I} \subseteq \mathcal{K}$ . Therefore  $d^n([M] \otimes 1) = \pm \gamma_n(\zeta'')$ . Now comparing coefficients in equations (6) and (7), the result follows.  $\square$

#### 4. PROOF OF THEOREM 1.1

We now have everything required to prove Theorem 1.1 via a routine Serre spectral sequence argument. Let  $\mathcal{E} = \{\mathcal{E}^r, \delta^r\}$  be the homology Serre spectral sequence for the free loop space fibration sequence

$$\Omega M \xrightarrow{\vartheta} \mathcal{L}M \xrightarrow{ev_1} M.$$

By Proposition 3.1 we have an isomorphism  $H_*(\Omega M) \cong U = T(V)/I$  of Hopf algebras, which are free as  $R$ -modules. So we start with an isomorphism of free  $R$ -modules

$$\mathcal{E}_{*,*}^2 \cong R\{1, a_1, \dots, a_m, [M]\} \otimes U.$$

By Proposition 2.2

$$\delta^n(a_i \otimes y) = -1 \otimes [u_i, y]$$

where  $u_i$  is the transgression of  $a_i$ , and using (6),

$$\delta^n([M] \otimes y) = - \sum_{i,j} c_{ij} (a_j \otimes [u_i, y]).$$

Therefore  $\mathcal{E}_{0,*}^{2n} \cong \mathcal{Q}$ ,  $\mathcal{E}_{n,*}^\infty \cong \mathcal{E}_{n,*}^{2n} \cong \mathcal{W}$ , and  $\mathcal{E}_{2n,*}^{2n} \cong \mathcal{Z}$ , while all other entries in the spectral sequence are zero. Here, the only possible nonzero differentials are  $\delta^{2n} : \mathcal{E}_{2n,*}^{2n} \longrightarrow \mathcal{E}_{0,*+2n-1}^{2n}$ . But since the nonzero elements in  $\mathcal{Z}$  and  $\mathcal{Q}$  are concentrated in total degrees  $2n + k(n-1)$  and  $k(n-1)$  respectively, one can check the differentials  $\delta^{2n}$  are zero for degree placement reasons whenever  $n > 2$ . Thus these isomorphisms carry over to the infinity page, that is,

$$\mathcal{E}_{*,*}^\infty \cong \mathcal{E}_{0,*}^\infty \oplus \mathcal{E}_{n,*}^\infty \oplus \mathcal{E}_{2n,*}^\infty \cong \mathcal{Q} \oplus \mathcal{W} \oplus \mathcal{Z}.$$

Generally, one has torsion here when  $R = \mathbb{Z}$  (or at least in  $\mathcal{Q}$ , and possibly  $\mathcal{W}$ ), so we must consider a potential extension problem. Once again placement reasons allow us to skirt around the issue.

From the construction of the homology Serre spectral sequence there are increasing filtrations  $\mathcal{F}_{i,j} = \mathcal{F}_i H_j(\mathcal{L}M) \subseteq H_j(\mathcal{L}M)$  such that  $\mathcal{F}_{k,k} = H_k(\mathcal{L}M)$ ,  $\mathcal{F}_{i,j} = 0$  for  $i < 0$ , and

$$\mathcal{E}_{i,j}^\infty \cong \frac{\mathcal{F}_{i,i+j}}{\mathcal{F}_{i-1,i+j}}.$$

Since the nonzero elements in  $\mathcal{Q}$ ,  $\mathcal{W}$ , and  $\mathcal{Z}$  are in degrees  $k(n-1)$ ,  $n+k(n-1)$ , and  $2n+k(n-1)$ ,  $\mathcal{Q}$ ,  $\mathcal{W}$ , and  $\mathcal{Z}$  pairwise have no nonzero elements in the same degrees when  $n > 3$ . Since  $\mathcal{F}_{n-1,*} = \mathcal{F}_{0,*} = \mathcal{Q}$ , we have  $\mathcal{F}_{n-1,n+k(n-1)} = \{0\}$ , and we see that  $\mathcal{F}_{n,*} \cong \mathcal{F}_{0,*} \oplus \mathcal{E}_{n,*}^\infty \cong \mathcal{Q} \oplus \mathcal{W}$ . Then  $\mathcal{F}_{2n-1,2n+k(n-1)} = \mathcal{F}_{n,2n+k(n-1)} = \{0\}$ , so  $\mathcal{F}_{2n,*} \cong \mathcal{F}_{n,*} \oplus \mathcal{E}_{2n,*}^\infty$ , and we have

$$\mathcal{E}_{2n,*}^\infty \cong \mathcal{F}_{2n,*} = H_*(\mathcal{L}M)$$

whenever  $n > 3$ .



When  $n = 3$ , the common nonzero degrees shared between any pair of these three modules are of the form  $2(k+3)$ , and these are only between  $\mathcal{Q}$  and  $\mathcal{Z}$ . But since  $\mathcal{Z}$  is torsion-free and  $\mathcal{Q} = \mathcal{F}_{0,*}$  is at the bottom of the filtration, there are no extension issues here either.

## 5. EILENBERG-MACLANE SPACES AND THE BV-OPERATOR

We will need some information about the action of the BV-operator on products of Eilenberg-MacLane spaces before getting into the proof Theorem 1.2. The approach we take here is similar to the one taken by Hepworth in [18] to compute the BV-operator for Lie groups. We begin this section by recalling it. Fix  $R$  to be a principal ideal domain, and  $X$  (homotopy type of a  $CW$ -complex) a path-connected topological group with multiplication  $X \times X \xrightarrow{\mu} X$ . This makes  $\mathcal{L}X$  into topological group with multiplication  $\mathcal{L}X \times \mathcal{L}X \xrightarrow{\mathcal{L}\mu} \mathcal{L}X$  defined by point-wise multiplication of loops  $(\omega \cdot \gamma)(t) = \omega(t) \cdot \gamma(t)$ . There is a well-known homeomorphism

$$\begin{aligned} h: X \times \Omega X &\longrightarrow \mathcal{L}X \\ h(x, \omega) &= x \cdot \omega \end{aligned}$$

with inverse  $h^{-1}: \mathcal{L}X \longrightarrow X \times \Omega X$  given by  $h^{-1}(\omega) = (\omega(0), \omega(0)^{-1} \cdot \omega)$ , where  $x \cdot \omega$  is the loop defined at each point by  $(x \cdot \omega)(t) = x \cdot \omega(t)$ . These homeomorphisms are equivariant with respect to our action  $S^1 \times \mathcal{L}X \xrightarrow{\nu} \mathcal{L}X$ , and the action

$$\bar{\nu}: S^1 \times X \times \Omega X \longrightarrow X \times \Omega X$$

defined by the formula

$$\bar{\nu}(t, x, \omega) = h^{-1} \circ \nu(t, x \cdot \omega) = (x \cdot \omega_t(0), (x \cdot \omega_t(0))^{-1} \cdot x \cdot \omega_t) = (x \cdot \omega_t(0), \omega_t(0)^{-1} \cdot \omega_t)$$

where  $\omega_t(s) = \nu(t, \omega)(s) = \omega(s+t)$ . On homology we have a commutative square

$$\begin{array}{ccc} H_*(X \times \Omega X; R) & \xrightarrow[\cong]{h_*} & H_*(\mathcal{L}X; R) \\ \bar{\Delta} \downarrow & & \downarrow \Delta \\ H_{*+1}(X \times \Omega X; R) & \xrightarrow[\cong]{h_*} & H_{*+1}(\mathcal{L}X; R). \end{array}$$

where  $\bar{\Delta}(e) = \bar{\nu}_*([S^1] \otimes e)$ . Clearly, after transposing  $X$  and  $S^1$ ,  $\bar{\nu}$  is the composite

$$X \times (S^1 \times \Omega X) \xrightarrow{1_X \times \Delta} X \times (S^1 \times \Omega X) \times (S^1 \times \Omega X) \xrightarrow{1_X \times ev \times \phi} (X \times X) \times \Omega X \xrightarrow{\mu \times 1} X \times \Omega X,$$

with  $ev: S^1 \times \Omega X \longrightarrow X$  the evaluation map  $ev(t, \omega) = \omega(t) = \omega_t(0)$ , and  $\phi: S^1 \times \Omega X \longrightarrow \Omega X$  defined by  $\phi(t, \omega) = \omega_t(0)^{-1} \cdot \omega_t$ . Thus, if  $H_*(\Omega X; R)$  is a free  $R$ -module, so that (for simplicity) the cross product  $H_*(X; R) \otimes H_*(\Omega X; R) \xrightarrow{\times} H_*(X \times \Omega X; R)$  is an isomorphism, and the coproduct on an element  $b \in H_*(\Omega X; R)$  has the form  $\Delta_*(b) = \sum_i d_i \otimes e_i$ , then  $\bar{\Delta}$  satisfies

$$\begin{aligned} (8) \quad (-1)^{|a|} \bar{\Delta}(a \otimes b) &= \sum_i (-1)^{|d_i|} (a(ev_*(1 \otimes d_i)) \otimes \phi_*([S^1] \otimes e_i)) + \sum_i (a(ev_*([S^1] \otimes d_i)) \otimes \phi_*(1 \otimes e_i)) \\ &= \sum_i (-1)^{|d_i|} (a\epsilon(d_i) \otimes \phi_*([S^1] \otimes e_i)) + \sum_i (a(ev_*([S^1] \otimes d_i)) \otimes e_i). \end{aligned}$$

where  $\epsilon: H_*(\Omega X; R) \longrightarrow R$  is the augmentation. To complete this formula one needs to determine the maps  $\phi_*$  and  $ev_*$ . This latter map defines the *homology suspension*  $\sigma: H_*(\Omega X; R) \longrightarrow H_{*+1}(X; R)$ ,  $\sigma(a) = ev_*([S^1] \otimes a)$ , which satisfies the formula

$$(9) \quad \sigma(ab) = \sigma(a)\epsilon(b) + \epsilon(a)\sigma(b)$$

for any product  $ab \in H_*(\Omega X; R)$  induced by the loop composition multiplication on  $\Omega X$ . In particular,  $\sigma$  is zero on decomposable elements. If  $X$  is an  $H$ -space, one can derive this formula by observing that the following diagram commutes

$$\begin{array}{ccccc} (S^1 \times S^1) \times (\Omega X \times \Omega X) & \xrightarrow{\mathbb{1} \times T \times \mathbb{1}} & (S^1 \times \Omega X) \times (S^1 \times \Omega X) & \xrightarrow{ev \times ev} & X \times X \\ \Delta \times \mathbb{1} \times \mathbb{1} \uparrow & & & & \downarrow \mu \\ S^1 \times (\Omega X \times \Omega X) & \xrightarrow{\mathbb{1} \times \Omega \mu} & S^1 \times \Omega X & \xrightarrow{ev} & X, \end{array}$$

and that point-wise multiplication of based loops  $\Omega \mu$  on  $\Omega X$  is homotopy commutative and homotopic to the loop composition multiplication on  $\Omega X$  (this is a mapping space analogue of Theorem 5.21, Chapter III in [27]). Alternatively, it is a consequence of the Homology Suspension Theorem ([27], Chapter VIII). The map  $\kappa(a) = \phi_*([S^1] \otimes a)$  is a bit more mysterious. At the very least, when  $\mu$  is commutative one obtains an analogous commutative diagram for  $\phi$  together with a derivation formula  $\kappa(ab) = \kappa(a)b + a\kappa(b)$ , while for the case of compact Lie groups,  $\kappa$  is trivial since  $H_*(\Omega X)$  is concentrated even degrees. We consider the case where  $X$  is an Eilenberg-MacLane space  $K(R, n)$ . These can be taken to be commutative topological groups, and we may write  $K(G, n-1) = \Omega K(G, n)$  with commutative multiplication induced by the one on  $K(R, n)$ , which by the way is homotopic to the loop composition multiplication.

**Proposition 5.1.** *Let  $J$  be the image of the cross product  $H_*(K(R, n-1); R) \otimes H_*(K(R, n); R) \xrightarrow{\times} H_*(K(R, n-1) \times K(R, n); R)$  (which is injective by the Künneth formula). Suppose the coproduct on  $a \in H_*(K(R, n-1); R)$  is in the image of the cross product, that is, it is of the form  $\Delta_*(b) = \sum_i d_i \times e_i$ . Then with respect to the isomorphism  $h_*$ , the BV-operator is given on  $a \times b \in J \subseteq H_*(\mathcal{L}K(R, n); R)$  by the formula*

$$\Delta(a \times b) = (-1)^{|a|} \sum_i (a(\rho_*([S^1] \otimes d_i)) \times e_i),$$

where  $\Sigma K(R, n-1) \xrightarrow{\rho} K(R, n)$  is a classifying map for  $[S^1] \otimes \iota_{n-1} \in H^*(\Sigma K(R, n-1); R) \cong \bar{H}^*(S^1; R) \otimes \bar{H}^*(K(R, n-1); R)$ , and  $\iota_{n-1} \in \bar{H}^{n-1}(K(R, n-1); R)$  is the fundamental class.

*Proof.* Since our map  $S^1 \times K(R, n-1) \xrightarrow{\phi} K(R, n-1)$  restricts to the identity on the right factor,  $\phi^*(\iota_{n-1}) = 1 \otimes \iota_{n-1}$ , or in other words,  $\phi$  is a classifying map of the cohomology class  $1 \otimes \iota_{n-1} \in \bar{H}^{n-1}(S^1 \times K(R, n-1); R)$ . The projection map onto the right factor  $S^1 \times K(R, n-1) \xrightarrow{* \times \mathbb{1}} K(R, n-1)$  is also a classifying map for  $1 \otimes \iota_{n-1}$ . Since cohomology classes are in one-to-one correspondance with the homotopy classes of the classifying maps representing them,  $\phi$  must be homotopic to  $* \times \mathbb{1}$ . Therefore  $\phi_*([S^1] \otimes d) = 0$  for any  $d$ .

Next, recall the suspension isomorphism  $H_{n-1}(K(R, n-1); R) \xrightarrow{\cong} H_n(\Sigma K(R, n-1); R)$ , sending  $a \mapsto [S^1] \otimes a$ , factors as the composite

$$H_{n-1}(K(R, n-1); R) \xrightarrow{\cong} [K(R, n-1), K(R, n-1)] \xrightarrow{\cong} [\Sigma K(R, n-1), K(R, n)]$$

where the last map is the adjoint isomorphism. Since the evaluation map  $S^1 \times K(R, n-1) \xrightarrow{ev} K(R, n)$  restricts to the constant map on both the left and right factors, it factors as the composite

$$ev: S^1 \times K(R, n-1) \xrightarrow{\text{quotient}} \Sigma K(R, n-1) \xrightarrow{ev'} K(R, n),$$

where the last map  $ev'$  (also known as the evaluation map in the literature) is the adjoint of the identity map  $K(R, n-1) \xrightarrow{\mathbb{1}} K(R, n-1)$ . Since the identity is a classifying map of  $\iota_{n-1}$ , by the above factorization of the suspension, its adjoint  $ev'$  is a classifying map of  $[S^1] \otimes \iota_{n-1}$ . The proposition now follows using equation (8). □

The BV-operator has a very clean form on decomposable elements when we take our multiplication on  $H_*(\mathcal{L}X)$  to be the one induced by point-wise multiplication of loops  $\mathcal{L}\mu$  (instead of the multiplication  $(\Omega\mu \times \mu) \circ (\mathbb{1} \times T \times \mathbb{1})$  based on each coordinate of  $\Omega X \times X \cong \mathcal{L}X$ ). Tamanoi [25] gave a derivation formula with respect to this product

$$\Delta(ab) = \Delta(a)b + (-1)^{|a|}a\Delta(b),$$

which is a straightforward consequence of the following commutative diagram

$$\begin{array}{ccccc} (S^1 \times S^1) \times (\mathcal{L}X \times \mathcal{L}X) & \xrightarrow{\mathbb{1} \times T \times \mathbb{1}} & (S^1 \times \mathcal{L}X) \times (S^1 \times \mathcal{L}X) & \xrightarrow{\nu \times \nu} & \mathcal{L}X \times \mathcal{L}X \\ \Delta \times \mathbb{1} \times \mathbb{1} \uparrow & & & & \downarrow \mathcal{L}\mu \\ S^1 \times (\mathcal{L}X \times \mathcal{L}X) & \xrightarrow{\mathbb{1} \times \mathcal{L}\mu} & S^1 \times \mathcal{L}X & \xrightarrow{\nu} & \mathcal{L}X. \end{array}$$

Both multiplications on  $\mathcal{L}X$  are equal when the multiplication on  $X$  is commutative. Since this is the case for  $K(R, n)$ , our formula in Proposition 5.1 satisfies

$$(10) \quad (-1)^{|b||c|} \Delta(ac \times bd) = \Delta((a \times b)(c \times d)) = \Delta(a \times b)(c \times d) + (-1)^{|a|+|b|}(a \times b)\Delta(c \times d)$$

The derivation formula can also be used to determine how the BV-operator interacts with the cross-product, as we see in the following:

**Proposition 5.2.** *Let  $X = X_1 \times \cdots \times X_k$  be a product of topological groups  $(X_i, \mu_i)$ . Then the BV-operator for  $\mathcal{L}X \cong \mathcal{L}X_1 \times \cdots \times \mathcal{L}X_k$  satisfies*

$$\Delta(a_1 \times \cdots \times a_k) = \sum_i (-1)^{k_i} (a_1 \times \cdots \times \Delta(a_i) \times \cdots \times a_k)$$

for  $a_i \in H_*(\mathcal{L}X_i)$ , where  $k_i = \sum_{j=1}^{i-1} |a_j|$  and  $k_1 = 0$ .

*Proof.* It suffices to prove the statement for length-2 products  $X = X_1 \times X_2$ . One can then iterate to obtain the general formula. Since the inclusion of the left factor  $\mathcal{L}X_1 \xrightarrow{\mathbb{1} \times * } \mathcal{L}X_1 \times \mathcal{L}X_2$  induces the map on homology sending  $a \mapsto a \times 1$  for any  $a$ , by naturality of the BV-operator we have  $\Delta(a_1 \times 1) = (\mathbb{1} \times *)_*(\Delta(a_1)) = \Delta(a_1) \times 1$ . Similarly,  $\Delta(1 \times a_2) = 1 \times \Delta(a_2)$ . Since  $X$  is a topological group with multiplication  $\mu$  defined by the composite  $X \times X \xrightarrow{\mathbb{1} \times T \times \mathbb{1}} (X_1 \times X_1) \times (X_2 \times X_2) \xrightarrow{\mu_1 \times \mu_2} X$ , the point-wise loop multiplication  $\mathcal{L}\mu$  is the composite

$$\mathcal{L}X \times \mathcal{L}X \xrightarrow{\cong} (\mathcal{L}X_1 \times \mathcal{L}X_2) \times (\mathcal{L}X_1 \times \mathcal{L}X_2) \xrightarrow{\mathbb{1} \times T \times \mathbb{1}} (\mathcal{L}X_1 \times \mathcal{L}X_1) \times (\mathcal{L}X_2 \times \mathcal{L}X_2) \xrightarrow{\mathcal{L}\mu_1 \times \mathcal{L}\mu_2} \mathcal{L}X.$$

Therefore  $(a_1 \times 1)(1 \times a_2) = a_1 \times a_2$  with respect to this induced product, and by the derivation formula we have

$$\begin{aligned} \Delta(a_1 \times a_2) &= \Delta(a_1 \times 1)(1 \times a_2) + (-1)^{|a_1|}(a_1 \times 1)\Delta(a_2 \times 1) \\ &= \Delta(a_1) \times a_2 + (-1)^{|a_1|}a_1 \times \Delta(a_2). \end{aligned}$$

□

We have, for the sake of simplicity, been restricting  $X$  to be a topological group. Some of the material above however extends (up-to-homotopy) to where  $X$  is a homotopy associative  $H$ -space. In this scenario  $h$  is a homotopy equivalence since it defines a weak equivalence between the free loop fibration and the trivial fibration. If  $X$  has an inverse  $-\mathbb{1}: X \rightarrow X$ ,  $x \mapsto x^{-1}$ , the null homotopy  $H: X \times X \times I \rightarrow X$ , with  $H_0 = *$  and  $H_1 = \mathbb{1} \times -\mathbb{1}$ , allows us to define the homotopy inverse  $h^{-1}$  just as before, except this time composing the loop  $\omega(0)^{-1} \cdot \omega$  with the based path given by  $H_t(\omega(0)^{-1}, \omega(0))$ , and the action  $\bar{\nu}$  will have a similar form.

In the case of rational coefficients, a simply connected  $H$ -space  $X$  has a rational decomposition  $X_{\mathbb{Q}} \simeq \prod_i K(\mathbb{Q}, n_i)$ , and the classifying maps  $\Sigma K(\mathbb{Q}, n_i - 1) \rightarrow K(\mathbb{Q}, n_i)$  can be identified with the Freudenthal suspension  $S_{\mathbb{Q}}^{n_i} \rightarrow \Omega \Sigma S_{\mathbb{Q}}^{n_i}$  in the  $n_i$  even case, and evaluation  $\Sigma \Omega S_{\mathbb{Q}}^{n_i} \rightarrow S_{\mathbb{Q}}^{n_i}$  in the

odd case. We see then that the action of  $\Delta$  on  $H_*(\mathcal{L}X; \mathbb{Q})$  with respect to the algebra structure induced by the group multiplication on  $\prod_i K(\mathbb{Q}, n_i)$  can be determined by applying Propositions 5.2 and 5.1.

This technique can still be used to obtain some useful information for more general coefficients. Suppose  $H_*(X; R)$  is free as an  $R$ -module, and  $a \in H_n(X; R)$  is an indecomposable element in the Hopf algebra  $H_*(X; R)$ . Then the cohomology dual  $\hat{a} \in H^n(X; R)$  of  $a$  is a primitive element in the dual Hopf algebra  $H^*(X; R)$ , the classifying map  $X \xrightarrow{c} K(R, n)$  of  $\hat{a}$  is an  $H$ -map, and moreover it is natural with respect to the homeomorphism  $h$ . That is, the following squares commute up to homotopy

$$(11) \quad \begin{array}{ccc} X \times X & \xrightarrow{c \times c} & K(R, n) \times K(R, n) \\ \mu \downarrow & & \downarrow \text{mult.} \\ X & \xrightarrow{c} & K(R, n) \end{array} \quad \begin{array}{ccc} X \times \Omega X & \xrightarrow{c \times \Omega c} & K(R, n) \times \Omega K(R, n) \\ h \downarrow \cong & & \cong \downarrow h \\ \mathcal{L}X & \xrightarrow{\mathcal{L}c} & \mathcal{L}K(R, n). \end{array}$$

The proof of commutativity is as follows. For degree reasons, the fundamental class  $\iota_n$  satisfies  $(\text{mult.})^*(\iota_n) = (\iota_n \times 1 + 1 \times \iota_n)$ , so we have  $(c \times c)^* \circ (\text{mult.})^*(\iota_n) = \hat{a} \otimes 1 + 1 \otimes \hat{a}$ . Likewise, since  $\hat{a}$  is primitive,  $\mu^* \circ c^*(\iota_n) = \mu^*(\hat{a}) = \hat{a} \otimes 1 + 1 \otimes \hat{a}$ . Thus both the composites in the first square are classifying maps of  $\hat{a} \otimes 1 + 1 \otimes \hat{a}$ , meaning they are homotopic. This gives the first square. To obtain the second square, let  $H: (X \times X) \times I \rightarrow K(R, n)$  be a choice of homotopy between the composites in the first square. Define the homotopy  $G: (X \times \Omega X) \times I \rightarrow \mathcal{L}K(R, n)$  by  $G(x, \omega, t) = \omega_{x,t}$ , where  $\omega_{x,t}: S^1 \rightarrow X$  is the loop given by  $\omega_{x,t}(s) = H(x, \omega(s), t)$ . Then  $G$  defines a homotopy between the two composites in the second square. As a consequence of these diagrams,  $\mathcal{L}c_*$  is an algebra map with respect to the algebra structure induced by the isomorphisms  $h_*$ , given by  $(\mathcal{L}c)_*(v \otimes b) = c_*(v) \times (\Omega c)_*(b)$ .

Now suppose  $n$  is odd,  $a$  is transgressive, and  $\tau(a) \in H_{n-1}(\Omega X; R)$  is its transgression. Since  $c_*$  maps  $a$  to the homology dual  $\hat{\iota}_n$  of  $\iota_n$ , and  $\hat{\iota}_n$  is transgressive onto  $\tau(\hat{\iota}_n) = \hat{\iota}_{n-1}$ , the homology dual of the fundamental class of  $\Omega K(R, n) = K(R, n-1)$ , we have  $(\Omega c)_*(\tau(a)) = \hat{\iota}_{n-1}$ . Then  $(\mathcal{L}c)_*(\Delta(v \otimes \tau(a))) = \Delta((\mathcal{L}c)_*(v \otimes \tau(a))) = \Delta(c_*(v) \times \hat{\iota}_{n-1}) = (-1)^{|v|} (c_*(v) \hat{\iota}_{n-1}) \times 1$  by Proposition 5.1, and applying the derivation formula (10) inductively,

$$(\mathcal{L}c)_*(\Delta(v \otimes \tau(a)^k)) = \Delta(c_*(v) \otimes \hat{\iota}_{n-1}^k) = k(-1)^{|v|} ((c_*(v) \hat{\iota}_{n-1}) \times \hat{\iota}_{n-1}^{k-1}).$$

Since  $(\mathcal{L}c)_*(va \otimes \tau(a)^{k-1}) = (c_*(v) \hat{\iota}_n) \times \hat{\iota}_{n-1}^{k-1}$ , if we assume  $\tau(a)^{k-1}$  generates  $H_{(k-1)(n-1)}(\Omega X; R)$ , and  $va$  generates  $H_{n+|v|}(X; R)$ , then

$$\Delta(v \otimes \tau(a)^k) = k(-1)^{|v|} (va \otimes \tau(a)^{k-1}).$$

For example, if we take  $R = \mathbb{Z}_p$  for  $p$  an odd prime,  $X = S_{(p)}^n$  as a  $p$ -localized sphere (which is an  $H$ -space for  $n$  odd [1]), and  $a = [S^n]$ , then this formula completely determines the action of  $\Delta$  on  $H(\mathcal{L}S^n; \mathbb{Z}_p) \cong H(\mathcal{L}X; \mathbb{Z}_p)$ . This is a somewhat different approach for spheres than the one taken by Westerland in [26], and Menichi in [21].

## 6. PROOF OF THEOREM 1.2

For degree placement reasons, it is clear that  $\Delta(\mathcal{Q}) \subseteq \mathcal{W}$ ,  $\Delta(\mathcal{W}) \subseteq \mathcal{Z}$ , and  $\Delta(\mathcal{Z}) = \{0\}$  when  $n > 3$ . Consider the composite

$$f: M \xrightarrow{\Delta} \prod_{i=1}^m M \xrightarrow{\prod_i f_i} \prod_{i=1}^m K(\mathbb{Q}, n) = P,$$

where  $f_i$  is the classifying map of the generator  $a_i \in H^n(M; \mathbb{Q})$ . Let  $\iota_i \in H_n(K(\mathbb{Q}, n); \mathbb{Q})$  denote the homology dual of the fundamental class for the  $i^{\text{th}}$  factor, and  $\bar{\iota}_i \in H_{n-1}(K(\mathbb{Q}, n-1); \mathbb{Q})$  the corresponding transgression. Let  $W = \mathbb{Q}\{\iota_1, \dots, \iota_m\}$  and  $\bar{W} = \mathbb{Q}\{\bar{\iota}_1, \dots, \bar{\iota}_m\}$ .

Since  $n$  is odd,  $H_*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[\iota_i]$ ,  $H_*(K(\mathbb{Q}, n-1); \mathbb{Q}) \cong \mathbb{Q}[\bar{\iota}_i]$ ,  $f$  induces the injection  $H_*(M; \mathbb{Q}) \cong V \oplus K \rightarrow \Lambda_{\mathbb{Q}}[W]$ , mapping  $a_i \mapsto \iota_i$  and  $[M] \mapsto \beta = \sum_{i < j} (c_{ij} \iota_i \iota_j)$ , and  $\Omega f$  induces the algebra map  $\mathcal{Q} \xrightarrow{\eta_q} \mathbb{Q}[\bar{W}] \cong S(V)$ , mapping  $u_i \mapsto \bar{\iota}_i$ .

Consider the morphism of rational homology Serre spectral sequences  $\mathcal{E} \xrightarrow{\phi} E$  induced by the map of free loop space fibrations

$$\begin{array}{ccccc} \Omega M & \longrightarrow & \mathcal{L}M & \xrightarrow{ev_1} & M \\ \downarrow \Omega f & & \downarrow \mathcal{L}f & & \downarrow f \\ \Omega P & \longrightarrow & \mathcal{L}P & \xrightarrow{ev_1} & P. \end{array}$$

The spectral sequence  $E$  for the bottom fibration collapses since the total space is a topological group with section. On the infinity page

$$H_*(\mathcal{L}P; \mathbb{Q}) \cong H_*(P; \mathbb{Q}) \otimes H_*(\Omega P; \mathbb{Q}) \cong \bigoplus_{i=0}^m E_{ni,*}^{\infty},$$

and  $\phi^{\infty}$  restricts to the maps  $\mathcal{Q} \xrightarrow{\eta_q} \mathbb{Q}[\bar{W}] \cong E_{0,*}^{\infty}$ ,  $\mathcal{W} \xrightarrow{\eta_w} W \otimes \mathbb{Q}[\bar{W}] \cong E_{n,*}^{\infty}$ , and  $\mathcal{Z} \xrightarrow{\eta_z} \mathbb{Q}\{\beta\} \otimes \mathbb{Q}[\bar{W}] \subseteq E_{2n,*}^{\infty}$  (note  $W \cong V$ ,  $\mathbb{Q}\{\beta\} \cong K$ , and  $\mathbb{Q}[\bar{W}] \cong S(V)$  in the introduction).

Let  $F$  be the filtration of  $H_*(\mathcal{L}P; \mathbb{Q})$  associated with the spectral sequence  $E$ . Notice  $E_{n,*}^{\infty} \cong F_{n,n+*}/\mathbb{Q}[\bar{W}]$ , and  $\mathbb{Q}[\bar{W}]$  is concentrated in degrees  $k(n-1)$ , while  $\mathcal{W}$  is concentrated in degrees  $n+k(n-1)$ , which are never equal when  $n > 3$ , so they do not share any nonzero elements in the same degree. Similarly,  $E_{2n,*}^{\infty} \cong F_{2n,2n+*}/F_{n,2n+*}$ ,  $F_{n,*} \cong \mathbb{Q}[\bar{W}] \oplus (W \otimes \mathbb{Q}[\bar{W}])$  is concentrated in degrees  $k(n-1)$  and  $n+k(n-1)$ , and  $\mathcal{Z}$  is concentrated in degrees  $2n+k(n-1)$ , which are never equal when  $n > 3$ . Therefore, with respect to our isomorphism  $H_*(\mathcal{L}M; \mathbb{Q}) \cong \mathcal{Q} \oplus \mathcal{W} \oplus \mathcal{Z}$ ,  $(\mathcal{L}f)_*$  restricts to the maps  $\eta_q$ ,  $\eta_w$ , and  $\eta_z$  on each summand.

The action of  $\Delta$  on  $H_*(\mathcal{L}K(\mathbb{Q}, n-1); \mathbb{Q})$  is given by  $\Delta(1 \otimes \bar{\iota}_i^k) = k(\iota_i \otimes \bar{\iota}_i^{k-1})$  and  $\Delta(a \otimes \bar{\iota}_i) = 0$  when  $|a| > 0$ . This follows from Proposition 5.1, and iterating formula (10). Alternatively, it follows from [26, 21]. Now by Proposition 5.2,

$$\Delta(a \otimes \bar{\iota}_1^{k_1} \dots \bar{\iota}_m^{k_m}) = \sum_{i=1}^m k_i (a \iota_i \otimes \bar{\iota}_i^{k_1} \dots \bar{\iota}_i^{k_i-1} \dots \bar{\iota}_m^{k_m}) \subseteq W \otimes \mathbb{Q}[\bar{W}] \cong A \otimes S(V)$$

for any integers  $k_i \geq 0$ . Since for any  $q \in \mathcal{Q}$ , we have  $\Delta(q) \in \mathcal{W}$ ,

$$\Delta \circ \eta_q(q) = \Delta \circ (\mathcal{L}f)_*(q) = (\mathcal{L}f)_* \circ \Delta(q) = \eta_w \circ \Delta(q),$$

we obtain the formula for the composite  $\mathcal{Q} \xrightarrow{\Delta} \mathcal{W} \xrightarrow{\eta_w} A \otimes S(V)$ . Similarly we obtain the formula for the composite  $\mathcal{W} \xrightarrow{\Delta} \mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V)$ .

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